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# First-order locally variational operators

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**Abstract**
First-order Lagrangian densities with first-order Euler–Lagrange operator are characterized. Variationally trivial Lagrangians and locally variational operators of first order are determined.

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## 1. Introduction

The inverse problem of the calculus of variations consists in characterizing those differential operators which are the Euler–Lagrange operator of a Lagrangian density. The goal of this paper is to provide a geometric characterization of the first-order Lagrangian densities whose Euler–Lagrange operator is of first order. Furthermore, this characterization allows us to determine the variationally trivial Lagrangians and the locally variational operators of first order.

## 2. First-order Euler–Lagrange operators

### *2.1. Notations and preliminaries*

Throughout this section we consider a fibred manifold $p\colon M \to N$ (i.e. $p$ is a surjective submersion of smooth manifolds) with $\dim N = n$, $\dim M = m + n$. Let $V(p) = V(M) = \{X \in T(M) : p_*X = 0\}$ be its vertical bundle. The global sections of $V(p)$ are denoted by $\mathfrak{X}^v(p)$, or else by $\mathfrak{X}^v(M)$. We also denote by $\mathfrak{X}^v_c(p)$ (or by $\mathfrak{X}^v_c(M)$) the ideal of vector fields $X \in \mathfrak{X}^v(p)$ such that $p(\text{support } X)$ is compact. Let $p_r\colon J^r(p) \to N$ (or even $J^rM$) be the $r$-jet bundle of local sections of $p$, and by $p_{rs}\colon J^r(p) \to J^s(p)$, $r \geqslant s$, the natural projections. If $(x^i; y^\alpha)$, $1 \leqslant i \leqslant n$, $1 \leqslant \alpha \leqslant m$, is a fibred coordinate system for $p$, then the induced coordinates on $J^r(p)$ are denoted by $\left(y_I^\alpha\right)$, $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $0 \leqslant |I| \leqslant r$, with $|I| = i_1 + \cdots + i_n$, $y_0^\alpha = y^\alpha$. The total derivative with respect to the variable $x^i$ is denoted by

$$\frac{\mathrm{d}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + \sum_{|I|\geqslant 0} y^\alpha_{I+(i)} \frac{\partial}{\partial y_I^\alpha}$$

where $(i) = (0, \ldots, 0, \overset{(i}{1}, 0, \ldots, 0) \in \mathbb{N}^n$. Let $\mathcal{C}^r(p)$ be the sheaf of germs of 1-contact differential forms. A vector field $X \in \mathfrak{X}(J^r M)$ is said to be an infinitesimal contact transformation (cf [8, 9]) if $L_X \mathcal{C}^r(p) \subseteq \mathcal{C}^r(p)$. We denote by $X^{(r)}$ the infinitesimal contact transformation associated with a vector field $X \in \mathfrak{X}(M)$. We also denote by (cf [6, section 4])

$$h\colon p^*_{r+1,r} \bigwedge^q T^* J^r M \to \bigwedge^q T^* J^{r+1} M$$
$$h\left(j_x^{r+1} s, \omega\right) = p^*_{r+1}((j^r s)^* \omega)$$

the horizontalization operator. If $\omega$ is a $q$-form on $J^r(p)$, the form $p^*_{r+1,r}\omega$ admits a decomposition as follows (e.g., see [6, section 4]): $p^*_{r+1,r}\omega = \sum_{i=0}^q h_i(\omega)$, where $h_i(\omega)$ is the $i$-contact component of $\omega$ (which is $(q-i)$ horizontal) and $h_0(\omega) = h(\omega)$.

We denote by $\mathcal{E}[\mathcal{L}]$ the Euler–Lagrange operator of a first-order Lagrangian density $\mathcal{L}$ on $p\colon M \to N$, which is considered as a section of the pull-back to $J^2(p)$ of the bundle $T^*M \wedge \bigwedge^n T^*N$; i.e. $\mathcal{E}[\mathcal{L}] \in \Gamma(J^2 M, T^*M \wedge \bigwedge^n T^*N)$. The operator $\mathcal{E}[\mathcal{L}]$ is said to be of order $s \leqslant 2$ if it can be obtained as the pull-back of a section in $\Gamma(J^s M, T^*M \wedge \bigwedge^n T^*N)$. The Euler–Lagrange operator is determined by the equation

$$\int_N (j^1 s)^* (X^{(1)} \lrcorner\, \mathrm{d}\mathcal{L}) = \int_N (j^2 s)^* (X \lrcorner\, \mathcal{E}[\mathcal{L}]) \tag{2.1}$$

for every open subset $U \subseteq N$, every section $s$ of $p$ defined on $U$ and every $X \in \mathfrak{X}^v_c(p^{-1}(U))$.

### *2.2. Densities and horizontalization*

**Proposition 2.1.** *The operator* $h_p\colon \Omega^{n+p}(M) \to \Omega^{n+p}(J^1 M)$ *is injective for every* $p \geqslant 0$.

**Proof.** We proceed by recurrence on $p$. For $p = 0$ we remark that $h_p$ coincides with the horizontalization operator. Let

$$\Lambda = \sum_{k=0}^n \sum_{j_1 < \cdots < j_k} \sum_{\alpha_1 < \cdots < \alpha_k} A^{\alpha_1 \ldots \alpha_k}_{j_1 \ldots j_k} \mathrm{d}y^{\alpha_1} \wedge \cdots \wedge \mathrm{d}y^{\alpha_k} \wedge \mathrm{vol}_{j_1 \ldots j_k} \tag{2.2}$$

be the local expression of $\Lambda \in \Omega^n(M)$, where

$$\mathrm{vol}_{j_1 \ldots j_k} = \partial/\partial x^{j_k} \lrcorner \ldots \lrcorner\, \partial/\partial x^{j_1}(\mathrm{vol}).$$

If we define

$$L_k = \sum_{j_1 < \cdots < j_k} \sum_{\alpha_1 < \cdots < \alpha_k} A^{\alpha_1 \ldots \alpha_k}_{j_1 \ldots j_k} y^{\alpha_1}_{j_1} \ldots y^{\alpha_k}_{j_k}$$

then we have $h(\Lambda) = L\mathrm{vol}$, with $L = \sum_{k=0}^n L_k$. As $L_k$ is the homogeneous component of degree $k$ of $L$, we have

$$A^{\alpha_1 \ldots \alpha_k}_{j_1 \ldots j_k} = \frac{\partial^k L_k}{\partial y^{\alpha_1}_{j_1} \ldots \partial y^{\alpha_k}_{j_k}}. \tag{2.3}$$

Hence $\Lambda \in \ker h$ if and only if $L_k = 0$, and then, $A^{\alpha_1 \ldots \alpha_k}_{j_1 \ldots j_k} = 0$; that is $\Lambda = 0$.

Assume $p > 0$. Let us fix an index $1 \leqslant \alpha \leqslant m$. For every $\omega \in \Omega^{n+p}(M)$ we have $\omega = \mathrm{d}y^\alpha \wedge \omega_\alpha + \eta_\alpha$ (no summation), where $\partial/\partial y^\alpha \lrcorner\, \omega_\alpha = 0$, $\partial/\partial y^\alpha \lrcorner\, \eta_\alpha = 0$. Hence $h_p(\omega) = \mathrm{d}y^\alpha \wedge h_{p-1}(\omega_\alpha) + \eta'_\alpha$, with $\partial/\partial y^\alpha \lrcorner\, \eta'_\alpha = 0$. If $h_p(\omega) = 0$, then the coefficient of $\mathrm{d}y^\alpha$ must vanish; hence $h_{p-1}(\omega_\alpha) = 0$, and by virtue of the recurrence hypothesis, we conclude $\omega_\alpha = 0$ for every index $\alpha$, and we can finish the proof. □

**Theorem 2.2.** *Let $\mathcal{L}$ be a first-order Lagrangian density. The following conditions are equivalent:*

*(a) There exists $\Lambda \in \Omega^n(M)$ such that $\mathcal{L} = h(\Lambda)$.*

*(b) The Euler–Lagrange operator $\mathcal{E}[\mathcal{L}]$ is of first order.*

*If these conditions hold, then $\mathcal{E}[\mathcal{L}] = h_1(\mathrm{d}\Lambda)$.*

**Proof.** If $\mathcal{L} = h(\Lambda)$, then $p_{10}^*\Lambda = \mathcal{L} + \theta$, where $\theta$ is a 1-contact form. For every $X \in \mathfrak{X}_c^v(M)$ we have

$$L_{X^{(1)}}\mathcal{L} = X^{(1)} \lrcorner\, p_{10}^*(\mathrm{d}\Lambda) + \mathrm{d}(X^{(1)} \lrcorner\, p_{10}^*\Lambda) - L_{X^{(1)}}\theta. \tag{2.4}$$

As $\deg \mathrm{d}\Lambda = n + 1 > \dim N$, we have $h(\mathrm{d}\Lambda) = 0$ and $p_{10}^*(\mathrm{d}\Lambda) = h_1(\mathrm{d}\Lambda) + \theta'$, where $\theta'$ is a 2-contact form. Hence $X^{(1)} \lrcorner\, p_{10}^*(\mathrm{d}\Lambda) = X^{(1)} \lrcorner\, h_1(\mathrm{d}\Lambda) + X^{(1)} \lrcorner\, \theta'$, where $X^{(1)} \lrcorner\, \theta'$ is a 1-contact form. Substituting the right-hand side of this equation into (2.4) and recalling that $X$ is $p$-vertical, we obtain

$$X^{(1)} \lrcorner\, \mathrm{d}\mathcal{L} = L_{X^{(1)}}\mathcal{L} = X^{(1)} \lrcorner\, h_1(\mathrm{d}\Lambda) + \mathrm{d}(X^{(1)} \lrcorner\, p_{10}^*\Lambda) + X^{(1)} \lrcorner\, \theta' - L_{X^{(1)}}\theta.$$

Since $X^{(1)} \lrcorner\, \theta' - L_{X^{(1)}}\theta$ is a 1-contact form, pulling the previous equation back along $j^1s$, from Stokes' theorem and the formula (2.1) we obtain

$$\int_N (j^1s)^*(X^{(1)} \lrcorner\, \mathrm{d}\mathcal{L}) = \int_N (j^1s)^*(X \lrcorner\, h_1(\mathrm{d}\Lambda)) = \int_N (j^2s)^*(X \lrcorner\, \mathcal{E}[\mathcal{L}]).$$

Hence $\mathcal{E}[\mathcal{L}] = h_1(\mathrm{d}\Lambda)$. Moreover, the Euler–Lagrange operator

$$\mathcal{E}[\mathcal{L}] = \left( \frac{\partial L}{\partial y^\alpha} - \frac{\partial^2 L}{\partial x^j \partial y_j^\alpha} - \frac{\partial^2 L}{\partial y^\beta \partial y_j^\alpha} y_j^\beta - \frac{\partial^2 L}{\partial y_i^\beta \partial y_j^\alpha} y_{(ji)}^\beta \right) \mathrm{d}y^\alpha \wedge \mathrm{vol}$$

is of first order if and only if the following equations hold:

$$\frac{\partial^2 L}{\partial y_i^\beta \partial y_j^\alpha} + \frac{\partial^2 L}{\partial y_j^\beta \partial y_i^\alpha} = 0 \qquad \forall i, j. \tag{2.5}$$

Letting $i = j$ in (2.5), we deduce that $L$ is a polynomial of degree $n$ in the first derivatives with coefficients in $C^\infty(M)$; say $L = \sum A_{j_1 \ldots j_r}^{\alpha_1 \ldots \alpha_r} y_{j_1}^{\alpha_1} \ldots y_{j_r}^{\alpha_r}$, where we assume that $A_{j_1 \ldots j_r}^{\alpha_1 \ldots \alpha_r}$ is symmetric with respect to each of the pairs $\alpha_i, j_i$, for $i = 1, \ldots, r$. The Lagrangian $L$ satisfies equation (2.5) if and only if all its homogeneous components $L_k$ satisfy such an equation.

Substituting $L_k$ for $L$ into (2.5) and taking partial derivatives with respect to $y_{j_1}^{\alpha_1}, \ldots, y_{j_{k-2}}^{\alpha_{k-2}}$, we obtain

$$\frac{\partial^k L_k}{\partial y_{j_1}^{\alpha_1} \ldots \partial y_{j_{k-2}}^{\alpha_{k-2}} \partial y_i^\beta \partial y_j^\alpha} + \frac{\partial^k L_k}{\partial y_{j_1}^{\alpha_1} \ldots \partial y_{j_{k-2}}^{\alpha_{k-2}} \partial y_j^\beta \partial y_i^\alpha} = 0$$

and recalling formula (2.3), we have $A_{j_1 \ldots j_{k-2} ij}^{\alpha_1 \ldots \alpha_{k-2} \beta\alpha} + A_{j_1 \ldots j_{k-2} ji}^{\alpha_1 \ldots \alpha_{k-2} \beta\alpha} = 0$. Hence $A_{j_1 \ldots j_k}^{\alpha_1 \ldots \alpha_k}$ is skew-symmetric in the indices $j_1, \ldots, j_k$ (keeping $\alpha_1, \ldots, \alpha_k$ fixed), and conversely, that is $A_{\sigma(j_1) \ldots \sigma(j_k)}^{\alpha_1 \ldots \alpha_k} = (-1)^\sigma A_{j_1 \ldots j_k}^{\alpha_1 \ldots \alpha_k}$, for every permutation $\sigma$ of $\{j_1, \ldots, j_k\}$. Accordingly,

$$\begin{aligned} L_k \mathrm{vol} &= \sum_{\alpha_1, \ldots, \alpha_k} \sum_{j_1 < \cdots < j_k} \sum_\sigma \mathrm{sign}(\sigma) A_{j_1 \ldots j_k}^{\alpha_1 \ldots \alpha_k} y_{\sigma(j_1)}^{\alpha_1} \cdots y_{\sigma(j_k)}^{\alpha_k} \mathrm{vol} \\ &= h\left( \sum_{\alpha_1, \ldots, \alpha_k} \sum_{j_1 < \cdots < j_k} A_{j_1 \ldots j_k}^{\alpha_1 \ldots \alpha_k} \mathrm{d}y^{\alpha_1} \wedge \cdots \wedge \mathrm{d}y^{\alpha_k} \wedge \mathrm{vol}_{j_1, \ldots, j_k} \right). \end{aligned}$$

□

**Remark 2.3.** In the conditions of theorem 2.2, we have $\int (j^1 s)^* \mathcal{L} = \int s^* \Lambda$, and hence the extremals of a variational problem are characterized by the Hamilton–Cartan equations for $\Lambda$

$$s^*(X \lrcorner \mathrm{d}\Lambda) = 0 \qquad \forall X \in \mathfrak{X}(M). \tag{2.6}$$

As a consequence of theorem 2.2 and proposition 2.1, we obtain a well-known characterization of first-order variationally trivial Lagrangians (see [5, p 36]).

**Corollary 2.4.** *For a first-order Lagrangian density $\mathcal{L}$ on $p\colon M \to N$, the two conditions below are equivalent:*

*(a) $\mathcal{L}$ is variationally trivial.*
*(b) There exists a closed form $\Lambda \in \Omega^n(M)$ such that $\mathcal{L} = h(\Lambda)$.*

**Proof.** If (a) holds, $\mathcal{E}[\mathcal{L}] = 0$; hence from theorem 2.2 we have $\mathcal{L} = h(\Lambda)$, where $\Lambda$ is an $n$-form on $M$. As $\mathcal{E}[\mathcal{L}] = h_1(\mathrm{d}\Lambda)$ and the operator $h_1$ is injective by virtue of proposition 2.1, we conclude. Conversely, if (b) holds, then we obtain $\mathcal{E}[\mathcal{L}] = \mathcal{E}[h(\Lambda)] = h_1(\mathrm{d}\Lambda) = 0$. □

**Remark 2.5.** From the previous corollary (and the well-known fact that $\Omega^r(M)$ is generated as a $C^\infty(M)$-module by closed $r$-forms), we obtain another equivalent condition to those in theorem 2.2:

(c) The Lagrangian density $\mathcal{L}$ is a finite sum $f_i \mathcal{L}^i$, where $f_i \in C^\infty(M)$, and every $\mathcal{L}^i$ is a first-order variationally trivial Lagrangian density.

**Remark 2.6.** As an immediate consequence of theorem 2.2, we also obtain the characterization of variational $(n+1)$-forms on $J^1 M$ stated in [7, theorem 1]; namely, an $(n+1)$-form $\tau$ on $J^1 M$ is variational, if and only if there exists an $n$-form $\Lambda$ on $M$ such that $\tau = \mathcal{E}[h(\Lambda)]$.

**Remark 2.7.** Let

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial y_i^\alpha} \theta^\alpha \wedge \mathrm{vol}_i + \mathcal{L}$$

be the Poincaré–Cartan form attached to the Lagrangian density $\mathcal{L} = L\mathrm{vol}$, where $\theta^\alpha = \mathrm{d}y^\alpha - y_j^\alpha \,\mathrm{d}x^j$ is the standard contact form. We know (e.g., see [4]) that $\Theta_{\mathcal{L}}$ is $p_{10}$-projectable onto $M$ if and only if the Lagrangian $L$ is an affine function over $p_{10}\colon J^1 \to M$; that is locally we should have $L = A_\alpha^i y_i^\alpha + A^0$ for certain functions $A^0, A_\alpha^i \in C^\infty(M)$. If this is the case, then $\mathcal{E}[\mathcal{L}]$ is of first order, but the converse is not true except for mechanics; i.e. when $n = \dim N = 1$. In the field theory, there are important examples of Lagrangians defined by affine functions. For example, if $N$ admits a spin structure, then the Dirac Lagrangian is an affine function. This explains why the equation for the free Dirac electron fields is of first order. Similarly, the scalar curvature density on the bundle of metrics of prescribed signature, $M$, of the ground manifold $N$ determines a second-order Lagrangian that is an affine function over $p_{21}\colon J^2 M \to J^1 M$.

An example of a Lagrangian density whose Euler–Lagrange operator is of first order, but which is not affine, is as follows: consider a $\sigma$-model defined by mappings $f\colon N \to F$, which corresponds to the sections of the trivial bundle $M = N \times F \to N$. We set $\mathcal{L}\big(j_x^1 f\big) = (f^*\omega)(x)$, where $\omega \in \Omega^n(F)$ is an arbitrary $n$-form. For example, the topological locally defined (in the sense of section 2.3) Wess–Zumino terms are obtained in this way, e.g., see [1, p 4].

**Remark 2.8.** Theorem 2.2 cannot directly be generalized to the higher order Lagrangians. Indeed, let $\mathcal{L}$ be the second-order Lagrangian on $p\colon \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\mathcal{L} = \begin{vmatrix} y^1_{11} & y^2_{11} & y^3_{11} \\ y^1_{12} & y^2_{12} & y^3_{12} \\ y^1_{22} & y^2_{22} & y^3_{22} \end{vmatrix} \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

It can be shown that $\mathcal{E}[\mathcal{L}]$ is a third-order operator, but it is clear that $\mathcal{L}$ cannot be written as $h(\Lambda)$ for any $\Lambda \in \Omega^n(J^1(p))$, as $\mathcal{L}$ is a polynomial of third degree in the derivatives of second order, while $h(\Lambda)$ is of second degree at most.

### 2.3. *Locally variational first-order operators*

**Proposition 2.9.** *A section $T \in \Gamma(J^1(p), \bigwedge^n T^*N \wedge T^*M)$ is a locally variational first-order operator if and only if there exists a closed form $\Omega_T \in \Omega^{n+1}(M)$ such that $T = h_1(\Omega_T)$. The form $\Omega_T$ is uniquely determined by T.*

**Proof.** Suppose $T$ is locally variational. As is well known (see [3], theorem 3.1), locally, there exists a Lagrangian $\mathcal{L}$ of order not greater than the order of $T$ (of first order, in this case) such that $\mathcal{E}[\mathcal{L}] = T$. By virtue of theorem 2.2, there exists a form $\Lambda \in \Omega^n(M)$ such that $\mathcal{L} = h(\Lambda)$; we define $\Omega_T = \mathrm{d}\Lambda$. According to corollary 2.4, if $\mathcal{L}' = h(\Lambda')$ is another first-order Lagrangian satisfying these conditions, then we have $h_1(\mathrm{d}\Lambda) = h_1(\mathrm{d}\Lambda')$; hence $\mathrm{d}\Lambda' = \mathrm{d}\Lambda = \Omega_T$, as $h_1$ is injective. So, $\Omega_T$ is a well-defined global form, and it is closed since it is locally exact. In addition, we have $T = h_1(\Omega_T)$, as follows from theorem 2.2.

Conversely, if $\Omega_T \in \Omega^{n+1}(M)$ is closed, it is locally exact, and locally there is $\Lambda \in \Omega^n(M)$ such that $\Omega_T = \mathrm{d}\Lambda$. From theorem 2.2 we have $h_1(\Omega_T) = \mathcal{E}[h(\Lambda)]$; i.e. $h_1(\Omega_T)$ is locally variational.

As for the uniqueness of $\Omega_T$, if $h_1(\Omega) = 0$ then $\Omega = 0$. □

**Remark 2.10.** The previous results can be summarized by saying that, in the following diagram, the vertical maps are isomorphisms:

$$\begin{array}{ccccc}
\left\{\begin{matrix}\text{closed}\\ n\text{-forms in } M\end{matrix}\right\} & \hookrightarrow & \{n\text{-forms in } M\} & \xrightarrow{d} & \left\{\begin{matrix}\text{closed}\\ (n+1)\text{-forms}\\ \text{in } M\end{matrix}\right\} \\
h\downarrow & & h\downarrow & & h_1\downarrow \\
\left\{\begin{matrix}\text{variationally}\\ \text{trivial}\\ \text{first-order}\\ \text{Lagrangians}\end{matrix}\right\} & \hookrightarrow & \left\{\begin{matrix}\text{first-order}\\ \text{Lagrangians}\\ \text{with first-order}\\ \text{Euler–Lagrange}\\ \text{operator}\end{matrix}\right\} & \xrightarrow{\mathcal{E}} & \left\{\begin{matrix}\text{locally variational}\\ \text{first-order}\\ \text{operators}\end{matrix}\right\}.
\end{array}$$

**Remark 2.11.** In the global version of the inverse problem in the calculus of variations (see [2, 10]), it is proved that the map

$$\Psi\colon H^{n+1}(M) \longrightarrow H^{n+1}(\mathcal{E}^*(J^\infty M)) \qquad \Psi([\omega]) = [(p_{\infty 1})^* h_1(\omega)]$$

is an isomorphism between the de Rham cohomology of $M$ and the cohomology of the Euler–Lagrange complex $\mathcal{E}^*(J^\infty M)$.

Using this property, we obtain the following results.

**Proposition 2.12.** *A closed form $\Omega \in \Omega^{n+1}(M)$ is exact if and only if $h_1(\Omega)$ is globally variational.*

**Proposition 2.13.** *If $\mathcal{L}$ is a Lagrangian density of order $r > 1$ with a first-order Euler–Lagrange operator, then there exists a first-order Lagrangian density $\mathcal{L}'$ such that $\mathcal{E}[\mathcal{L}'] = \mathcal{E}[\mathcal{L}]$.*

**Proposition 2.14.** *If $T$ is a locally variational operator, then there exists a Lagrangian density $\mathcal{L}$ and a first-order locally variational operator $T'$, such that*

$$T = \mathcal{E}[\mathcal{L}] + T'. \tag{2.7}$$

By virtue of the previous proposition, the study of the cohomological aspects of locally variational operators can be reduced to first-order ones. Note, however, that the descomposition (2.7) is not unique. For example, if $T$ has a symmetry, it is not sure that there exists a decomposition (2.7) in which $T'$ and $\mathcal{L}$ also have that symmetry.

If we define the set of extremal sections of a locally variational first-order operator $T = h_1(\Omega_T)$ by setting $\operatorname{Extr}(T) = \{s \in \Gamma(p) : T(j^1 s) = 0\}$, we have the corresponding Hamilton–Cartan equations, generalizing the equation (2.6):

$$\operatorname{Extr}(T) = \{s \in \Gamma(p) : s^*(X \lrcorner\, \Omega_T) = 0, \forall X \in \mathfrak{X}^v(p)\}. \tag{2.8}$$

**Example 2.15.** In [1] the author considers some examples of problems in which the Lagrangian density decomposes as $\mathcal{L} = \mathcal{L}_0 + \mathcal{T}$, where $\mathcal{L}_0$ is a globally well-defined Lagrangian and the term $\mathcal{T}$—the so-called topological term—satisfies the following conditions:

1. $\mathcal{T}$ may be interpreted as a differential form.
2. Under an appropriate transformation, $\mathcal{T}$ changes by a total derivation.
3. $\mathcal{T}$ is not globally defined.

We will show that this decomposition is related to that in (2.7), and the topological term is associated with a locally variational first-order operator. We will analyse the following example dealt with in [1]: consider a point particle moving in a two-dimensional Riemann surface $\Sigma$ in the presence of a magnetic monopole with potential $A_\mu \, \mathrm{d}x^\mu$. The classical Lagrangian for this system is

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + A_\mu \dot{x}^\mu.$$

The corresponding topological term is $\mathcal{T} = A_\mu \dot{x}^\mu \, \mathrm{d}t$. Integrating, we have $\int A_\mu \dot{x}^\mu \, \mathrm{d}t = \int_\Gamma A_\mu \, \mathrm{d}x^\mu$. In our setting, we can consider the fibred manifold $\mathbb{R} \times \Sigma \to \mathbb{R}$ and the form $A = \pi_2^*(A_\mu \, \mathrm{d}x^\mu)$. It is clear that $h(A) = \mathcal{T}$ and, hence, it corresponds to a first-order Lagrangian with first-order Euler–Lagrange equations. The problem is that $A$ is not a globally defined 1-form, it is only locally defined. (Remark that $\mathbf{A}$ is a 1-form on the corresponding principal bundle, and in a local chart can be represented by a 1-form on $\Sigma$.) If we consider two different local charts, the corresponding forms $A$ and $A'$ will differ in a gauge transformation: $A - A' = \mathrm{d}f$; but the form $F = \mathrm{d}A$ (the curvature) is a globally well-defined 2-form in $\Sigma$.

This can be explained as follows: $F$ represents a first-order locally variational operator that will not be globally variational unless $[F] = 0$ in $H^2(\Sigma, \mathbb{R})$, and the topological term in the Lagrangian is a 'Lagrangian density' for this operator. Also, if we consider the Euler–Lagrange operator of $\mathcal{L}$, we obtain $\mathcal{E}[\mathcal{L}_0] + T_F$, as in the decomposition (2.7). So, this system is described by a locally variational operator $T$, and its cohomology class in the adequate Euler–Lagrange complex is that corresponding to $[F]$.

In [1] the quantization condition of this system is shown to be that $\frac{1}{2\pi}[F]$ belongs to $H^2(\Sigma, \mathbb{Z})$; i.e. it imposes integrability conditions to the cohomology class of $T$ in the Euler–Lagrange complex.

Note, however, that in the topological term $\mathcal{T}$ there is more information than in $F$. For example, in [1] it is shown that, if quantization conditions are satisfied, then $\exp(\mathrm{i} \int_\Gamma A)$ is well defined.

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